

Linear Algebra in Any Dimension, Linear Systems and Matrices: Section 2 Test

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September 4, 2026

1 Introduction

In that test, we will prove the following result about the polynomials.

Theorem 1. *There is at most one polynomial function of degree $n - 1$ or less whose graph passes through n points in the plane with distinct x -coordinates.*

2 The maximum number zeros of a non zero polynomial

We will start with the example of a quadratic polynomial with 3 zeros, proving that it is the null polynomial.

Namely, we will prove that if a quadratic polynomial function $P(x) = a_0 + a_1x + a_2x^2$ is zero for $x = -1$, $x = 0$, and $x = 1$, then $a_0 = a_1 = a_2 = 0$.

2.1 Question 1.1: Substitute to x the values -1 , 0 , and 1 into $P(x)$

That is, calculate $P(-1)$, $P(0)$ and $P(1)$ as functions of a_0 , a_1 and a_2 .

2.2 Question 1.2: Set the results equal to 0

And obtain the system to solve in the variables a_0 , a_1 and a_2 .

2.3 Question 1.3: Solve the resulting system of linear equations in the variables a_0 , a_1 and a_2

TIP Find a_0 and add the 1st equation to the 3rd equation.

3 The polynomial functions which graph pass through given points

We may generalize the previous result the following way.

Theorem 2. *If a polynomial function $P(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$ is zero for n x -values or more, then $a_0 = a_1 = \dots = a_{n-1} = 0$.*

Consider n points of the plane $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$, with distinct abscissae x_i .

Consider the polynomials of degree less or equal to $n - 1$, $P(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$ and $Q(x) = b_0 + b_1x + \dots + b_{n-1}x^{n-1}$.

Assume that the graphs of the polynomial functions $P(x)$ and $Q(x)$ both pass through the points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$.

3.1 Question 2.1: The difference of the polynomial $P(x)$ and $Q(x)$

Consider the polynomial of degree less or equal than $n - 1$, $D(x) = c_0 + c_1x + \dots + c_{n-1}x^{n-1}$, where $c_i = a_i - b_i$ for $i = 0, 1, \dots, n - 1$.

Prove that for any real number $x \in \mathbb{R}$, $D(x) = P(x) - Q(x)$.

TIP Use the commutativity and associativity of the addition in $\mathbb{R}$, as well as the distributivity of the multiplication on the subtraction and of the '-' sign on the addition in $\mathbb{R}$.

3.2 Question 2.2: Prove that $x_1, x_2, \dots, x_n$ are zeros of the polynomial $D(x)$

TIP Calculate $P(x_i)$, $Q(x_i)$ and $D(x_i)$ for $i = 1, 2, \dots, n - 1$, using the question 2.1 for $D(x_i)$.

3.3 Question 2.3: Deduce that $P(x) = Q(x)$ from the theorem 2 and the question 2.2

TIP Prove that $a_i = b_i$ for $i = 1, 2, \dots, n - 1$, using the facts that $D(x)$ is of degree $n - 1$ or less and that the x_i are all distinct.

3.4 Conclusion

We have thus proved the theorem 1.

4 Conclusion

We have discovered the problem of polynomial interpolation of Lagrange. Indeed, if there is at most one polynomial function of degree $n - 1$ or less whose graph passes through n points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ in the plane with distinct x -coordinates, that polynomial exists and it is called the Lagrange polynomial for that list of points.

The Lagrange polynomial $L(x)$ is equal to $L(x) = \sum_{1 \leq j \leq n} y_j l_j(x)$, where

$$l_j(x) = \frac{(x-x_1) \dots (x-x_{j-1})(x-x_{j+1}) \dots (x-x_n)}{(x_j-x_1) \dots (x_j-x_{j-1})(x_j-x_{j+1}) \dots (x_j-x_n)} = \prod_{1 \leq i \leq n, i \neq j} \frac{x-x_i}{x_j-x_i}.$$

As each $l_j(x)$ has $n - 1$ factors of degree 1 at the numerator and a constant at the denominator, it is of degree $n - 1$. So that $L(x)$ is of degree less or equal to $n - 1$.

Moreover, for any j between 1 and n , $l_j(x_j) = 1$, then for any j , $L(x_j) = y_j$ and thus $L(x)$ solves the problem of polynomial interpolation of the list of points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$.